A combinatorial algorithm to compute presentations of mapping-class groups of orientable surfaces with one boundary component

Lluís Bacardit*

Abstract

We give an algorithm which computes a presentation for a subgroup, denoted $\mathcal{AM}_{g,1,p}$, of the automorphism group of a free group. It is known that $\mathcal{AM}_{g,1,p}$ is isomorphic to the mapping-class group of an orientable genus-g surface with one boundary component and p punctures. We define a variation of Auter space.

2010 *Mathematics Subject Classification.* Primary: 57N05, 20F05; Secondary: 20F28, 20F34.

Key words. Mapping-class groups, presentations, automorphism groups, Auter space.

1 Introduction

Let S be an orientable genus-g surface with b boundary components and p punctures. We denote by $\mathcal{M}(S)$ the group of isotopy classes of orientation-preserving homeomorphisms of S which permute the set of punctures and pointwise fix the boundary components. Since the group $\mathcal{M}(S)$ only depends, up to isomorphism, on the genus g of S, the number b of boundary components of S and the number p of punctures of S, we denote $\mathcal{M}(S)$ by $\mathcal{M}_{g,b,p}$. We call $\mathcal{M}_{g,b,p}$ the *mapping-class group* of S.

Presentations for $\mathcal{M}_{g,b,p}$ were obtained after a sequence of papers started by Hatcher and Thurston [11], and followed by Harer [10], Wajnryb [17],[18]; Matsumoto [14] and, Labruère and Paris [13]. For $p = 0$, Gervais [9] used [17] to deduce another presentations for $\mathcal{M}_{g,b,0}$. Before [11] very little was known about

*The research was funded by MIC (Spain) through Project MTM2008-01550.

the presentation of $\mathcal{M}_{g,b,p}$. Birman and Hilden [4] gave a presentation for $\mathcal{M}_{2,0,0}$, and McCool [15] proved that $\mathcal{M}_{g,b,p}$ is finitely presented.

Benvenuti [3] uses a variation of the curve complex, called ordered curve complex, to obtain presentations for $\mathcal{M}_{g,b,p}$ from an inductive process. This inductive process starts from presentation for the sphere and the torus with "few" boundary components and/or punctures. Hirose [12] uses the curve complex and induction on g and b to deduce Gervais presentation. Both of these papers are independent of [11].

Our algorithm is independent of [11]. We feel that our point of view goes back to McCool [15]. Section 7 contains the presentation given by our algorithm. This presentation has generators ze_i, ze_ie_j where z ranges over a finite set $\mathcal{L}$ and e_i, e_j range over z. There are three type of relations:

(a). $ze_i = 1$, $ze_{i'}e_{j'} = 1$, for some generators ze_i, $ze_{i'}e_{j'}$.

(b). $z_1e_ie_j = z_2e_{i'}e_{j'}$, for some generators $z_1e_ie_j$, $z_2e_{i'}e_{j'}$.

(c). $ze_i \cdot ze_ie_j = ze_j \cdot ze_je_i$, for every generator ze_ie_j.

Armstrong, Forrest and Vogtmann [1] give a new presentation for $\mathrm{Aut}(F_n)$, the automorphism group of the free group of rank n. This presentation for $\mathrm{Aut}(F_n)$ is obtained by studying the action of $\mathrm{Aut}(F_n)$ on a subcomplex of the spine of Auter space. Following Armstrong, Forrest and Vogtmann [1], we obtain our algorithm by studying the action of an algebraic analogous of $\mathcal{M}_{g,1,p}$ on a subcomplex of the spine of a variation of Auter space.

2 Preliminaries

Throughout the paper n will be a non-negative integer, F_n will be a free group of rank n, $\mathrm{Aut}(F_n)$ will be the automorphism group of F_n and $\mathrm{Out}(F_n)$ will be the automorphism group of F_n modulo inner automorphisms. Given $v, w \in F_n$, we denote by $[v, w]$ the element $v^{-1}w^{-1}vw$ of F_n. We denote by $[w]$ the conjugacy class of w.

Let S be an orientable genus-g surface with b boundary components and p punctures. A homeomorphism f of S which fixes the basepoint of $\pi_1(S)$ and permutes the set of punctures of S induces an automorphism $f_* \in \mathrm{Aut}(\pi_1(S))$. The isotopy class of f defines an automorphism of $\pi_1(S)$ up to inner automorphisms, that is, an element of $\mathrm{Out}(\pi_1(S))$. For $(b, p) = (0, 0)$, by Dehn-Nielsen-Baer Theorem, $\mathcal{M}_{g,0,0}$ is isomorphic to a index 2 subgroup of $\mathrm{Out}(\pi_1(S))$. For $(g, p) \neq (0, 0)$ by a modification of Dehn-Nielsen-Baer Theorem, $\mathcal{M}_{g,b,p}$ is isomorphic to an infinite index subgroup of $\mathrm{Out}(\pi_1(S))$.

Suppose now $b = 1$, that is, S has exactly one boundary component. If we choose the basepoint of $\pi_1(S)$ to be a boundary point of S and we restrict

ourselves to homeomorphisms of S which pointwise fix the boundary, then the isotopy class of a homeomorphism of S defines an element of $\mathrm{Aut}(\pi_1(S))$. Since S has one boundary component, the fundamental group of S is a free group. We denote by

$$\Sigma_{g,1,p} = \langle x_1, y_1, x_2, y_2, \ldots, x_g, y_g, t_1, t_2, \ldots, t_p \mid \; \rangle$$

a presentation of $\pi_1(S, *)$ where $*$ is a boundary point of S, for every $1 \leq k \leq p$ the generator t_k represents a loop around the k-th puncture of S and the word $[x_1, y_1][x_2, y_2] \cdots [x_g, y_g]t_1t_2 \cdots t_p$ represents a loop around the boundary component of S.

2.1 Definition. We denote by $\mathcal{AM}_{g,1,p}$ the subgroup of $\mathrm{Aut}(\Sigma_{g,1,p})$ consisting of automorphisms of $\Sigma_{g,1,p}$ which fix the word $[x_1, y_1][x_2, y_2] \cdots [x_g, y_g]t_1t_2 \cdots t_p$ of $\Sigma_{g,1,p}$ and fix the set of conjugacy classes $[t_1^{-1}], [t_2^{-1}], \ldots, [t_p^{-1}]$ of $\Sigma_{g,1,p}$. □

Using a modification of Dehn-Nielsen-Baer Theorem, it can be proved that $\mathcal{M}_{g,1,p}$ is isomorphic to $\mathcal{AM}_{g,1,p}$, see [8] with some changes of notation and some different conventions. We call $\mathcal{AM}_{g,1,p}$ the *algebraic mapping-class group* of an orientable genus-g surface with one boundary component and p punctures.

3 Auter space $\mathbb{A}_n$

3.1 Definition. Let (Γ, v_0, ϕ) be a 3-tuple such that

1. Γ is a finite connected graph with no separating edges.
2. Γ is a metric graph with total volume 1.
3. v_0 is a distinguished vertex of Γ.
4. Every vertex of Γ but v_0 has valence at least 3; v_0 has valence at least 2.
5. $\phi : \pi_1(\Gamma, v_0) \to F_n$ is an isomorphism called "marking".

A point in $\mathbb{A}_n$ is an equivalence class of 3-tuples (Γ, v_0, ϕ), where (Γ, v_0, ϕ) is equivalent to (Γ', v_0', ϕ') if there exists an isometry $h : \Gamma \to \Gamma$ such that $h(v_0) = v_0'$ and the isomorphism $h_* : \pi_1(\Gamma, v_0) \to \pi_1(\Gamma', v_0')$ satisfies $\phi = \phi' \circ h_*$.

We call $\mathbb{A}_n$ Auter space. □

Auter space $\mathbb{A}_n$ was introduced by Hatcher and Vogtmann [2] as an analogous for $\mathrm{Aut}(F_n)$ of Outer space. Often in the literature the marking is defined as $\phi^{-1} : F_n \to \pi_1(\Gamma, v_0)$.

If Γ has $k+1$ edges, then (Γ, v_0, ϕ) defines an open k-simplex of $\mathbb{A}_n$ denoted $\sigma(\Gamma, v_0, \phi)$. We can obtain $\sigma(\Gamma, v_0, \phi)$ by varying the length of the edges of Γ. The k-simplex $\sigma(\Gamma, v_0, \phi)$ can be parametrized by Δ^k, the standard open k-simplex of $\mathbb{R}^k$, as follows: $(\Gamma_s, v_0, \phi) \in \sigma(\Gamma, v_0, \phi)$ is the point of $\mathbb{A}_n$ such that

the length of the edges of Γ_s equal the barycentric coordinates of $s \in \Delta^k$. It is important that Δ^k is open. Since a non-trivial isometry of Γ permutes same edges of Γ, such an isometry gives a non-trivial element of $H_1(\Gamma)$. Hence, every $s \in \Delta$ defines a different point of $\sigma(\Gamma, v_0, \phi)$.

Some faces of $\sigma(\Gamma, v_0, \phi)$ belong to $\mathbb{A}_n$. If an edge of Γ is incident to two different vertices, then we can reduce the length of that edge to zero, and increase the length of the other edges, to obtain a new graph Γ' with one edge minus. We say that we have *collapsed* one edge of Γ. We have a quotient map $\Gamma \to \Gamma'$ which defines a point (Γ', v_0', ϕ') of $\mathbb{A}_n$. We say that $\sigma(\Gamma', v_0', \phi')$ is a face of $\sigma(\Gamma, v_0, \phi)$. Faces of $\sigma(\Gamma', v_0', \phi')$ are faces of $\sigma(\Gamma, v_0, \phi)$. We cannot collapse an edge which is incident to a unique vertex. Hence, some face of $\sigma(\Gamma, v_0, \phi)$ are missing. In particular, $\mathbb{A}_n$ is not a simplicial complex.

There exists a deformation retract, denoted $\mathrm{S}\mathbb{A}_n$, of $\mathbb{A}_n$ which is a simplicial complex. We can define $\mathrm{S}\mathbb{A}_n$ as follows. For every simplex of $\mathbb{A}_n$, there exists a vertex of $\mathrm{S}\mathbb{A}_n$. Two vertices of $\mathrm{S}\mathbb{A}_n$ expand an edge if the simplex of $\mathbb{A}_n$ which defines one of the two vertices of $\mathrm{S}\mathbb{A}_n$ is a face of the simplex of $\mathbb{A}_n$ which defines the other vertex of $\mathrm{S}\mathbb{A}_n$; $i+1$ vertices of $\mathrm{S}\mathbb{A}_n$ expand a i-simplex of $\mathrm{S}\mathbb{A}_n$ if every pair of vertices expand an edge.

There exists a natural inclusion of $\mathrm{S}\mathbb{A}_n$ into $\mathbb{A}_n$ by sending every vertex of $\mathrm{S}\mathbb{A}_n$ to the barycenter of the corresponding simplex, and every i-simplex of $\mathrm{S}\mathbb{A}_n$ to the convex hull of the corresponding barycenters. This inclusion is a deformation retract. See [2].

Collapsing an edge of Γ has an inverse process which *splits* a vertex of Γ into two new vertices, and, the two new vertices are joined by a new edge. Often in the literature splitting of a vertex is called blowing up an edge. If $\tilde{\Gamma}$ is obtained from Γ by splitting a vertex, then we can identify, in a natural way, every edge of Γ with an edge of $\tilde{\Gamma}$. Collapsing the only edge of $\tilde{\Gamma}$ which is not identified with an edge of Γ we obtain Γ. There exists a quotient map $\tilde{\Gamma} \to \Gamma$. If $\tilde{\Gamma}$ is obtained from Γ by splitting a vertex different from v_0, then the quotient map $\tilde{\Gamma} \to \Gamma$ defines a point $(\tilde{\Gamma}, \tilde{v}_0, \tilde{\phi})$ of $\mathbb{A}_n$ If $\tilde{\Gamma}$ is obtained from Γ by splitting v_0, then the point $(\tilde{\Gamma}, \tilde{v}_0, \tilde{\phi})$ of $\mathbb{A}_n$ depends of the election, between the two possibilities, of the new distinguished vertex $\tilde{v}_0$. The simplex $\sigma(\Gamma, v_0, \phi)$ is a face of $\sigma(\tilde{\Gamma}, \tilde{v}_0, \tilde{\phi})$.

We give a combinatorial definition of the topological type of Γ, that is, Γ when we forget its metric. When we forget the metric of Γ we can see (Γ, v_0, ϕ) as a vertex of $\mathrm{S}\mathbb{A}_n$, in fact, the vertex of $\mathrm{S}\mathbb{A}_n$ defined by the simplex $\sigma(\Gamma, v_0, \phi)$ of $\mathbb{A}_n$. We translate to our combinatorial definition the processes of collapsing an edges and splitting a vertex. Our combinatorial definition of the topological type of Γ is different from the one given in [6].

3.2 Definition. Let

1. $V(\Gamma)$ be the set of vertices of Γ.

2. $E(\Gamma)$ be the set of edges of Γ.

3. $\overline{E}(\Gamma) = \{\overline{e} \mid e \in E(\Gamma)\}$ be a set disjoint with $E(\Gamma)$.

We extend $^{-}$ to an involution of $E(\Gamma) \cup \overline{E}(\Gamma)$. We fix an orientation of every edge of Γ. We say that $e \in E(\Gamma)$ starts at $v_1 \in V(\Gamma)$ and finishes at $v_2 \in V(\Gamma)$ if e is incident to v_1 and v_2; and e is oriented from v_1 to v_2. In this case we say that $\overline{e}$ starts at v_2 and finishes at v_1.

Given $v \in V(\Gamma)$, we define the following subset of $E(\Gamma) \cup \overline{E}(\Gamma)$.

$$v^* = \{a \in E(\Gamma) \cup \overline{E}(\Gamma) \mid a \text{ starts at } v\}.$$

We set $V^*(\Gamma) = \{v^* \mid v \in V(\Gamma)\}$.

The topological type of Γ is completely determined by $(V(\Gamma), E(\Gamma), V^*(\Gamma))$. $\square$

Notice that v^* is the star of $v \in V(\Gamma)$ and $V^*(\Gamma)$ is a partition of $E(\Gamma)\cup\overline{E}(\Gamma)$. Condition 1 of Definition 3.1 can be translated by saying that $E(\Gamma)$ is finite and, for every $v \in V(\Gamma)$, there exist $a, b \in v^*$ such that $a \neq b, \overline{b}$ and $\overline{a}, \overline{b} \notin v^*$. Condition 4 of Definition 3.1 can be translated by saying that for every $v \in V(\Gamma) - \{v_0\}$, v^* has at least 3 elements; v_0^* has at least 2 elements.

3.3 Definition. Let $e \in E(\Gamma)$ such that $e \in v_1^*$, $\overline{e} \in v_2^*$, where $v_1, v_2 \in V(\Gamma)$ and $v_1 \neq v_2$. We can collapse e. When we collapse the edge e we have a graph with topological type

$$(V(\Gamma) \cup \{v\} - \{v_1, v_2\}, E(\Gamma) - \{e\}, V^*(\Gamma) \cup \{v^*\} - \{v_1^*, v_2^*\})$$

where $v \notin V(\Gamma)$ and $v^* = v_1^* \cup v_2^* - \{e, \overline{e}\}$. $\square$

3.4 Definition. Let $v \in V(\Gamma) - \{v_0\}$ and A, B a partition of v^* such that both A and B have at least two elements, there exists $a \in A$ such that $\overline{a} \notin A$ and there exists $b \in B$ such that $\overline{b} \notin B$. When we split the vertex v with respect to A and B we have a graph with topological type

$$(V(\Gamma) \cup \{v_1, v_2\} - \{v\}, E(\Gamma) \cup \{e\}, V^*(\Gamma) \cup \{v_1^*, v_2^*\} - \{v^*\}),$$

where $v_1, v_2 \notin V(\Gamma)$, $e \notin E(\Gamma)$, $v_1^* = A \cup \{e\}$ and $v_2^* = B \cup \{\overline{e}\}$. To split v_0 we have to choose between v_1 or v_2 as the new distinguished vertex. Since the distinguished vertex can have valence two, the subset of v_0^* corresponding to the new distinguished vertex may have only one element. $\square$

4 Ordered Auter space ord$\mathbb{A}_{g,p}$

Our motivation for defining ordered Auter space is that when a graph is embedded into an orientable surface, the star of every vertex of the graph which is mapped to an interior point of the surface gets a cyclic order, and, the star of a vertex which is mapped to a boundary point of the surface gets a linear order. When we want to collapse an edge or to split a vertex we have to do it respecting the orders of the stars.

4.1 Definition. Let $(\Gamma, v_0, \phi, \mathrm{ord})$ be a 4-tuple where (Γ, v_0, ϕ) satisfies conditions in Definition 3.1, ord is a linear order of v_0^* and a cyclic order of v^* for every $v \in V(\Gamma) - \{v_0\}$.

Suppose $V(\Gamma) = \{v_0, v_1, v_2, \ldots, v_q\}$ and

$$\begin{array}{rcl} \mathrm{ord}(v_0^*) & = & (a_1^0, a_2^0, \ldots, a_{r_0}^0), \\ \mathrm{ord}(v_1^*) & = & (a_1^1, a_2^1, \ldots, a_{r_1}^1), \\ \mathrm{ord}(v_2^*) & = & (a_1^2, a_2^2, \ldots, a_{r_2}^2), \\ & \vdots & \\ \mathrm{ord}(v_q^*) & = & (a_1^q, a_2^q, \ldots, a_{r_q}^q). \end{array} \tag{4.1.1}$$

For $i \neq 0$, since $\mathrm{ord}(v_i^*)$ is cyclically ordered, the subindices of $\mathrm{ord}(v_i^*)$ are modulo r_i

We consider the following element of $\pi_1(\Gamma, v_0)$ and the following conjugacy classes of $\pi_1(\Gamma, v_0)$.

$$\begin{array}{rcl} w_0 & = & b_1^0 b_2^0 \cdots b_{l_0}^0, \\ [w_1] & = & [b_1^1 b_2^1 \cdots b_{l_1}^1], \\ [w_2] & = & [b_1^2 b_2^2 \cdots b_{l_2}^2], \\ & \vdots & \\ [w_p] & = & [b_1^p b_2^p \cdots b_{l_p}^p], \end{array} \tag{4.1.2}$$

where $b_1^0 = a_1^0$, for every $1 \leq i \leq p$, $1 \leq j \leq l_i$ the subsequence $(\overline{b}_j^i, b_{j+1}^i)$ appears in (4.1.1), $b_{l_0}^0 = \overline{a}_{r_q}^q$, and every element of $E(\Gamma) \cup E(\Gamma)$ appears exactly once in (4.1.2).

We denote by $w(\Gamma, v_0, \mathrm{ord})$ the set $\{w_0, [w_1], [w_2], \ldots, [w_p]\}$. □

4.2 Example. Let $(\Gamma, v_0, \mathrm{ord})$ be a 3-tuple where $V(\Gamma) = \{v_0, v_1, v_2\}$, $E(\Gamma) = \{e_1, e_2, e_3, e_4, e_5\}$ and

$$\begin{array}{rcl} \mathrm{ord}(v_0^*) & = & (e_1, e_2), \\ \mathrm{ord}(v_1^*) & = & (\overline{e}_1, e_3, \overline{e}_3, e_4, e_5), \\ \mathrm{ord}(v_2^*) & = & (\overline{e}_2, \overline{e}_5, \overline{e}_4). \end{array}$$

Then $w(\Gamma, v_0, \mathrm{ord}) = \{w_0, [w_1], [w_2], [w_3]\}$ where

$$\begin{array}{rcl} w_0 & = & e_1 e_3 e_4 \overline{e}_2, \\ [w_1] & = & [\overline{e}_1 e_2 \overline{e}_5], \\ [w_2] & = & [\overline{e}_3], \\ [w_3] & = & [\overline{e}_4 e_5]. \end{array}$$

□

Notice that for every $v \in V(\Gamma)$, $\mathrm{ord}(v^*)$ is completely determined by $w(\Gamma, v_0, \mathrm{ord})$.

4.3 Definition. Let $(\Gamma, v_0, \phi, \mathrm{ord})$ be a 4-tuple such that $w(\Gamma, v_0, \mathrm{ord})$ has p conjugacy classes.

We denote $\frac{n-p}{2}$ by g. We will see that $n-p$ is even. Hence, g is a non-negative integer.

We define $\mathrm{ord}\mathbb{A}_{g,p}$ as the space of equivalence classes of 4-tuples $(\Gamma, v_0, \phi, \mathrm{ord})$ such that $\phi : \pi_1(\Gamma, v_0) \to \Sigma_{g,1,p}$, $w(\Gamma, v_0, \mathrm{ord}) = \{w_0, [w_1], [w_2], \ldots, [w_p]\}$ and

$$\begin{array}{rcl} \phi(w_0) & = & [x_1, y_1][x_2, y_2] \cdots [x_g, y_g] t_1 t_2 \cdots t_p, \\ \{\phi([w_1]), \phi([w_2]), \ldots, \phi([w_p])\} & = & \{[t_1^{-1}], [t_2^{-1}], \ldots, [t_p^{-1}]\}. \end{array}$$

The 4-tuples $(\Gamma, v_0, \phi, \mathrm{ord})$ and $(\Gamma', v_0', \phi', \mathrm{ord}')$ represent the same point of $\mathrm{ord}\mathbb{A}_{g,p}$ if there exists an isometry $h : \Gamma \to \Gamma'$ such that $h(v_0) = v_0'$, the isomorphism $h_* : \pi_1(\Gamma, v_0) \to \pi_1(\Gamma', v_0')$ satisfies $\phi = \phi' \circ h_*$, and $h : \Gamma \to \Gamma'$ preserves the orders, that is, $\mathrm{ord}(v^*) = (a_1, a_2, \ldots, a_r)$ implies $\mathrm{ord}'(h(v)^*) = (h(a_1), h(a_2), \ldots, h(a_r))$ for every $v \in V(\Gamma)$.

We call $\mathrm{ord}\mathbb{A}_{g,p}$ *ordered Auter space.*

We define $\mathrm{ordS}\mathbb{A}_{g,p}$ for $\mathrm{ord}\mathbb{A}_{g,p}$ as we defined $\mathrm{S}\mathbb{A}_n$ for $\mathbb{A}_n$. In particular, $\mathrm{ordS}\mathbb{A}_{g,p}$ is a simplicial complex, and, $\mathrm{ordS}\mathbb{A}_{g,p}$ is a deformation retract of $\mathrm{ord}\mathbb{A}_{g,p}$. □

The following definitions are based on Definition 3.3 and Definition 3.4, respectively.

4.4 Definition. Let $e \in E(\Gamma)$. Suppose $e = a^i_{k_1}$, $\overline{e} = a^j_{k_2}$, where $i \neq j$ and $1 \leq k_1 \leq r_i$, $1 \leq k_2 \leq r_j$. Since $i \neq j$, we can collapse e. We can suppose $j \neq 0$. To adapt Definition 3.3 to $\mathrm{ordS}\mathbb{A}_{g,p}$ we set

$$\begin{aligned} \mathrm{ord}(v^*) = (&a^i_{k-1}, a^i_{k_2}, \ldots, a^i_{k_1-1}, \\ &a^j_{k_2+1}, a^j_{k_2+2}, \ldots, a^j_{r_j}, a^j_1, a^j_2, \ldots, a^j_{k_2-1}, \\ &a^i_{k_1+1}, a^i_{k_1+2}, \ldots, a^i_{r_1}). \end{aligned}$$

□

4.5 Example. Let $(\Gamma, v_0, \mathrm{ord})$ be as in Example 4.2. When we collapse e_1 we obtain $(\Gamma', v_0', \mathrm{ord}')$ such that $v_0' = v_0$, $V(\Gamma') = \{v_0, v_2\}$, $E(\Gamma') = \{e_2, e_3, e_4, e_5\}$ and

$$\begin{array}{rcl} \mathrm{ord}'(v_0^*) & = & (e_3, \overline{e}_3, e_4, e_5, e_2), \\ \mathrm{ord}'(v_2^*) & = & (\overline{e}_2, \overline{e}_5, \overline{e}_4). \end{array}$$

We have $w(\Gamma', v_0', \mathrm{ord}') = \{w_0', [w_1'], [w_2'], [w_3']\}$ where

$$\begin{array}{rcl} w_0' & = & e_3 e_4 \overline{e}_2, \\ [w_1'] & = & [e_2 \overline{e}_5], \\ [w_2'] & = & [\overline{e}_3], \\ [w_3'] & = & [\overline{e}_4 e_5]. \end{array}$$

□

Let $(\Gamma, v_0, \phi, \mathrm{ord})$ be a vertex of $\mathrm{ordS}\mathbb{A}_{g,p}$. When we collapse an edge of Γ according to Definition 4.4 we obtain $(\Gamma', v_0', \phi', \mathrm{ord}')$. As it is see in Example 4.5, $w(\Gamma', v_0', \phi', \mathrm{ord}')$ has p conjugacy classes. Hence, $(\Gamma', v_0', \phi', \mathrm{ord}')$ is a vertex of $\mathrm{ordS}\mathbb{A}_{g,p}$.

4.6 Definition. Let $v \in V(\Gamma)$. Let A, B be a partition of v^*. Suppose $\mathrm{ord}(v^*) = (a_1, a_2, \dots, a_r)$ and $A = (a_{k_1}, a_{k_1+1}, \dots, a_{k_2})$, where $1 \leq k_1 < k_2 \leq r$. We can split the vertex v with respect to A, B. To adapt Definition 3.4 to $\mathrm{ordS}\mathbb{A}_{g,p}$ we set

$$\begin{aligned}\mathrm{ord}(v_1^*) &= (e, a_{k_1}, a_{k_1+1}, \dots, a_{k_2}),\\ \mathrm{ord}(v_2^*) &= (a_1, a_2, \dots, a_{k_1-1},\\ &\qquad \overline{e}, a_{k_2+1}, a_{k_2+2}, \dots a_r).\end{aligned}$$

□

4.7 Example. Let $(\Gamma, v_0, \mathrm{ord})$ be as in Example 4.2. When we split v_1 with respect to $\{\overline{e}_3, e_4\}$, $\{\overline{e}_1, e_3, e_5\}$ we obtain $(\tilde{\Gamma}, \tilde{v}_0, \tilde{\mathrm{ord}})$ such that $\tilde{v}_0 = v_0$, $V(\tilde{\Gamma}) = \{v_0, v_{1,1}, v_{1,2}, v_2\}$, $E(\tilde{\Gamma}) = \{e, e_1, e_2, e_3, e_4, e_5\}$ and

$$\begin{aligned}\tilde{\mathrm{ord}}(v_0^*) &= (e_1, e_2),\\ \tilde{\mathrm{ord}}(v_{1,1}^*) &= (e, \overline{e}_3, e_4),\\ \tilde{\mathrm{ord}}(v_{1,2}^*) &= (\overline{e}_1, e_3, \overline{e}, e_5),\\ \tilde{\mathrm{ord}}(v_2^*) &= (\overline{e}_2, \overline{e}_5, \overline{e}_4).\end{aligned}$$

We have $w(\tilde{\Gamma}, \tilde{v}_0, \tilde{\mathrm{ord}}) = \{\tilde{w}_0, [\tilde{w}_1], [\tilde{w}_2], [\tilde{w}_3]\}$ where

$$\begin{aligned}\tilde{w}_0 &= e_1 e_3 e_4 \overline{e}_2,\\ [\tilde{w}_1] &= [\overline{e}_1 e_2 \overline{e}_5],\\ [\tilde{w}_2] &= [\overline{e}_3 \overline{e}],\\ [\tilde{w}_3] &= [\overline{e}_4 e e_5].\end{aligned}$$

□

Let $(\Gamma, v_0, \phi, \mathrm{ord})$ be a vertex of $\mathrm{ordS}\mathbb{A}_{g,p}$. When we split a vertex of Γ according to Definition 4.6 we obtain $(\tilde{\Gamma}, \tilde{v}_0, \tilde{\phi}, \tilde{\mathrm{ord}})$. As it is see in Example 4.7, $w(\tilde{\Gamma}, \tilde{v}_0, \tilde{\phi}, \tilde{\mathrm{ord}})$ has p conjugacy classes. Hence, $(\tilde{\Gamma}, \tilde{v}_0, \tilde{\phi}, \tilde{\mathrm{ord}})$ is a vertex of $\mathrm{ordS}\mathbb{A}_{g,p}$.

Since a graph satisfying Definition 3.1 can have at most $3n - 2$ edges, the dimension of $\mathbb{A}_n$ is $3n - 3$. On the other hand, $\mathbb{A}_n$ is not a manifold. The dimension of $\mathrm{ord}\mathbb{A}_{g,p}$ is $6g + 3p - 3$; $\mathrm{ord}\mathbb{A}_n$ is a manifold.

By [2, PROPOSITION 2.1] $\mathbb{A}_n$ is contractible. Since $\mathrm{S}\mathbb{A}_n$ is a deformation retract of $\mathbb{A}_n$, we see that $\mathrm{S}\mathbb{A}_n$ is contractible.

4.8 Proposition. *ord*$\mathbb{A}_{g,p}$ *is contractible.*

Hatcher and Vogtmann proof [2, PROPOSITION 2.1] using spheres complexes. It is not clear how to translate to the context of spheres complexes an ordered graph. On the other hand, the proof of Culler and Vogtmann [6] that Outer space is contractible can be applied to $\mathrm{ord}\mathbb{A}_{g,p}$: adding a basepoint is straightforward, all the geometric arguments in [6] can be applied to $\mathrm{ord}\mathbb{A}_{g,p}$ respecting the orders as Definition 4.4 and Definition 4.6 and McCool [16], [7] proved that $\mathcal{AM}_{g,1,p}$ is generated by Nielsen automorphism which "respect" the orders (recall that Nielsen automorphisms are a special case of Whitehead automorphisms).

Recall $n = 2g+p$. There exists a natural map $\mathrm{ord}\mathbb{A}_{g,p} \to \mathbb{A}_n$ which "forgets" the ordering, that is, $(\Gamma, v_0, \phi, \mathrm{ord}) \mapsto (\Gamma, v_0, \phi)$. Recall that ord is completely determined by $w(\Gamma, v_0, \mathrm{ord})$. Since $\phi : \pi_1(\Gamma, v_0) \to \Sigma_{g,1,p}$ is an isomorphism, we have

$$\begin{aligned} w(\Gamma, v_0, \mathrm{ord}) = \ & \{\phi^{-1}([x_1, y_1][x_2, y_2] \cdots [x_g, y_g] t_1 t_2 \cdots t_p), \\ & [\phi^{-1}(t_1^{-1})], [\phi^{-1}(t_2^{-1})], \ldots, [\phi^{-1}(t_p^{-1})])\} \end{aligned}$$

Hence, the natural map $\mathrm{ord}\mathbb{A}_{g,p} \to \mathbb{A}_n$ is injective.

We want to see that $n - p$ is even.

For $n = 1$, we have $p = 1$ and $n - p = 0$ is even. We do induction on n.

By Definition 4.4 we can collapse a maximal subtree of $(\Gamma, v_0, \mathrm{ord})$. Hence, we can suppose that $V(\Gamma) = \{v_0\}$. Put $\mathrm{ord}(v_0^*) = (a_1, a_2, \ldots a_{2n})$ and $w(\Gamma, v_0, \mathrm{ord}) = \{w_0, [w_1], [w_2], \ldots, [w_p]\}$. Let $1 \leq j \leq 2n$ such that $a_j = \overline{a}_1$. Then $w_0 = a_1 u$ in reduced form, for some $u \in F_n$. Let $(\Gamma', v_0', \mathrm{ord}')$ be obtained from $(\Gamma, v_0, \mathrm{ord})$ be deleting the edges a_1, a_j. We have $\mathrm{ord}'(v_0') = (a_2, a_3, \ldots, a_{j-1}, a_{j+1}, a_{j+2}, \ldots, a_{2n})$. We put $n' = n - 1$ the rank of $\pi_1(\Gamma', v_0')$ and p' the number of conjugacy classes of $w(\Gamma', v_0', \mathrm{ord}')$. By induction hypothesis $n' - p'$ is even.

If $w_0 = a_1 u' a_j u''$ cyclically reduced, then $w(\Gamma', v_0', \mathrm{ord}') = \{u'', [u'], [w_1], [w_2], \ldots, [w_p]\}$. Notice that $u' \neq 1$, $u'' \neq 1$ because $a_1 u' a_j u''$ is cyclically reduced. Hence, $p' = p + 1$ and $n - p = (n' + 1) - (p' - 1) = n' - p' + 2$ is even.

If there exists $1 \leq k \leq p$ such that $[w_k] = [a_j w_k']$, then $w(\Gamma', v_0', \mathrm{ord}') = \{w_k' u', [w_1], [w_2], \ldots, [w_{k-1}], [w_{k+1}], \ldots, [w_p]\}$. Hence, $p' = p - 1$ and $n - p = (n' + 1) - (p' + 1) = n' - p'$ is even.

5 The Degree Theorem

Recall $\pi_1(\Gamma, v_0) \simeq F_n$.

We denote the valence of $v \in V(\Gamma)$ by $|v^*|$.

5.1 Definition. The degree of (Γ, v_0) is $2n - |v_0^*|$. Equivalently, the degree of (Γ, v_0) is $\sum_{v \in V(\Gamma)-\{v_0\}}(|v^*| - 2)$. □

To see the equivalence of the two definitions see [2, p. 636].

From the first definition of the degree of (Γ, v_0) we see that when we collapse an edge of Γ which is not incident with v_0 the degree is preserved, and, when we collapse an edge of Γ which is incident with v_0 the degree decreases. Hence, graphs of degree at most i expand a subcomplex D_i of $\mathrm{S}\mathbb{A}_n$. Hatcher and Vogtmann [2] proof the following.

5.2 Theorem. *D_i is i-dimensional and $(i-1)$-connected.*

In particular, D_2 is a simply-connected 2-complex.

We define $\mathrm{ord}D_i$ for $\mathrm{ordS}\mathbb{A}_{g,p}$ as we define D_i for $\mathrm{S}\mathbb{A}_n$.

All the arguments of Hatcher and Vogtmann to proof [2, THEOREM 3.3] can be applied to $\mathrm{ord}\mathbb{A}_{g,p}$. In particular, what they call "canonical splitting" and "sliding in the ϵ-cone" are combinations of splitting vertices and collapsing edges. We have the following.

5.3 Theorem. *$\mathrm{ord}D_i$ is i-dimensional and $(i-1)$-connected.*

In particular, $\mathrm{ord}D_2$ is a simply-connected 2-complex.

6 The action of $\mathcal{AM}_{g,1,p}$ on $\mathrm{ord}\mathbb{A}_n$

Recall that $\mathrm{Aut}(F_n)$ acts on $\mathbb{A}_n$ by "changing" the markings: for every $\varphi \in \mathrm{Aut}(F_n)$ we define $\varphi \cdot (\Gamma, v_0, \phi) = (\Gamma, v_0, \varphi \circ \phi)$. This action restricts to $\mathrm{S}\mathbb{A}_n$ and to D_2. The stabilizer of a vertex of $\mathrm{S}\mathbb{A}_n$ by this action is a finite group which permutes some edges and invert some edge orientations. The quotient complex $\mathrm{Aut}(F_n)\backslash\mathrm{S}\mathbb{A}_n$ is finite. See [1, Section 3], [2, Section 5]. Armstrong, Forrest and Vogtmann [1] apply a result of Brown [5] to $\mathrm{Aut}(F_n)\backslash D_2$ to compute a new presentation of $\mathrm{Aut}(F_n)$. Following this argument, we want to obtain a presentation of $\mathcal{AM}_{g,1,p}$.

We can define an action of $\mathcal{AM}_{g,1,p}$ on $\mathrm{ord}\mathbb{A}_{g,p}$ by "changing" the markings: for every $\varphi \in \mathcal{AM}_{g,1,p}$ we define $\varphi \cdot (\Gamma, v_0, \phi, \mathrm{ord}) = (\Gamma, v_0, \varphi \circ \phi, \mathrm{ord})$. This action restricts to $\mathrm{ordS}\mathbb{A}_{g,p}$ and to $\mathrm{ord}D_2$. The stabilizer of a vertex of $\mathrm{ordS}\mathbb{A}_{g,p}$ by this action is trivial and the quotient complex $\mathcal{AM}_{g,1,p}\backslash\mathrm{ordS}\mathbb{A}_{g,p}$ is finite, but much bigger than $\mathrm{Aut}(F_n)\backslash\mathrm{S}\mathbb{A}_n$. By Theorem 5.3, $\mathrm{ord}D_2$ is simply-connected. Hence, $\mathcal{AM}_{g,1,p}$ is isomorphic to the fundamental group of $\mathcal{AM}_{g,1,p}\backslash\mathrm{ord}D_2$. In the next section we give an algorithm which computes a presentation of the fundamental group of $\mathcal{AM}_{g,1,p}\backslash\mathrm{ord}D_2$.

7 The algorithm

Recall $n = 2g + p$.

Vertices of $\mathrm{ordS}\mathbb{A}_{g,p}$ are represented by 4-tuples $(\Gamma, v_0, \phi, \mathrm{ord})$ such that $w(\Gamma, v_0, \mathrm{ord})$ has p conjugacy classes. Recall that $\varphi \in \mathcal{AM}_{g,1,p}$ acts on $\mathrm{ordS}\mathbb{A}_{g,p}$ by "changing" the marking, that is, $\varphi \cdot (\Gamma, v_0, \phi, \mathrm{ord}) = (\Gamma, v_0, \varphi \circ \phi, \mathrm{ord})$. Hence, the quotient map $\mathrm{ordS}\mathbb{A}_{g,p} \to \mathcal{AM}_{g,1,p} \backslash \mathrm{ordS}\mathbb{A}_{g,p}$, $(\Gamma, v_0, \phi, \mathrm{ord}) \mapsto (\Gamma, v_0, \mathrm{ord})$ "forgets" the marking. We can represent vertices of $\mathcal{AM}_{g,1,p} \backslash \mathrm{ordS}\mathbb{A}_{g,p}$ by 3-tuples $(\Gamma, v_0, \mathrm{ord})$ such that $w(\Gamma, v_0, \mathrm{ord})$ has p conjugacy classes. Vertices of the subcomplex $\mathcal{AM}_{g,1,p} \backslash \mathrm{ord}D_2$ can be represented by 3-tuples $(\Gamma, v_0, \mathrm{ord})$ such that (Γ, v_0) has degree at most 2.

We want to compute a presentation for the fundamental group of complex $\mathcal{AM}_{g,1,p} \backslash \mathrm{ord}D_2$. Recall that the degree of (Γ, v_0) can be defined as $\sum_{v \in V(\Gamma) - \{v_0\}} (|v^*| - 2)$. Hence, if (Γ, v_0) has degree 2 then Γ has at most three vertices: v_0 and two more vertices of valence 3.

Let $\mathcal{L}$ be a list of vertices $(\Gamma, v_0, \mathrm{ord})$ of $\mathcal{AM}_{g,1,p} \backslash \mathrm{ord}D_2$ such that Γ has 3 vertices.

Let $z = (\Gamma, v_0, \mathrm{ord})$ be an element of $\mathcal{L}$. Suppose $E(\Gamma) = \{e_1, e_2, \ldots e_k\}$.

We construct a tree $T(z)$ as follows. There exists a vertex z of $T(z)$. Let e_i be an edge of Γ which can be collapsed, that is, e_i is incident to two different vertices. When we collapse e_i we have a quotient 3-tuple $z^i = (\Gamma^i, v_0^i, \mathrm{ord}^i)$. There exists a vertex z^i of $T(z)$ and an edge ze_i of $T(z)$ from z to z^i. We identify edges of z^i with edges of z. Let e_j be an edge of Γ^i which can be collapsed. When we collapse e_j in Γ^i we have a quotient 3-tuple $z^{(i,j)} = (\Gamma^{(i,j)}, v_0^{(i,j)}, \mathrm{ord}^{(i,j)})$. There exists a vertex $z^{(i,j)}$ of $T(z)$ and an edge ze_ie_j from z^i to $z^{(i,j)}$. We repeat this process for every edge which can be collapsed.

Our generating set for the fundamental group of $\mathcal{AM}_{g,1,p} \backslash \mathrm{ord}D_2$ is the set of edges of $T(z)$, where z ranges over $\mathcal{L}$.

The group $\mathrm{Sym}_k \times C_2^{\times k}$ acts on $E(\Gamma) \cup \overline{E}(\Gamma)$ by permuting edges ($C_2^{\times k}$ is the Cartesian product of k copies of the cyclic group of order 2). Hence, $\mathrm{Sym}_k \times C_2^{\times k}$ acts on the set of 3-tuples $(\Gamma, v_0, \mathrm{ord})$ by permuting edges and inverting edge orientations. Two 3-tuples $(\Gamma, v_0, \mathrm{ord})$ and $(\Gamma', v_0', \mathrm{ord}')$ represent the same vertex of $\mathcal{AM}_{g,1,p} \backslash \mathrm{ordS}\mathbb{A}_{g,p}$ if and only if they are in the same orbit by the action of $\mathrm{Sym}_k \times C_2^{\times k}$. Since every vertex of $T(z)$ is a 3-tuple $(\Gamma, v_0, \mathrm{ord})$, we see that $\mathrm{Sym}_k \times C_2^{\times k}$ acts on $T(z)$. We can identify $(\mathrm{Sym}_k \times C_2^{\times k}) \backslash T(z)$ with the 1-skeleton of a subcomplex of $\mathcal{AM}_{g,1,p} \backslash \mathrm{ord}D_2$. We can identify

$$(\mathrm{Sym}_k \times C_2^{\times k}) \backslash \Big(\bigcup_{z \in \mathcal{L}} T(z) \Big)$$

with the 1-skeleton of a subcomplex of $\mathcal{AM}_{g,1,p} \backslash \mathrm{ord}D_2$.

We attach some 2-cells to $(\mathrm{Sym}_k \times C_2^{\times k}) \backslash \big(\bigcup_{z \in \mathcal{L}} T(z) \big)$. If there exists the generator ze_ie_j, we attach a 2-cell though the egde-path $ze_i, ze_ie_j, \overline{ze_je_i}, \overline{ze_j}$. With

these attached 2-cells, the 2-complex $(\mathrm{Sym}_k \times C_2^{\times k})\backslash\big(\bigcup_{z\in\mathcal{L}} T(z)\big)$ is homeomorphic to $\mathcal{AM}_{g,1,p}\backslash \mathrm{ord}D_2$. We fix a maximal subtree of $(\mathrm{Sym}_k \times C_2^{\times k})\backslash\big(\bigcup_{z\in\mathcal{L}} T(z)\big)$.

Our presentation for the fundamental group of $\mathcal{AM}_{g,1,p}\backslash \mathrm{ord}D_2$ has three types of relations:

(a). $ze_i = 1$, $ze_{i'}e_{j'} = 1$ if the edges ze_i, $ze_{i'}e_{j'}$ are in our maximal subtree.

(b). $z_1e_ie_j = z_2e_{i'}e_{j'}$ if the generator $z_1e_ie_j$ exists and $g \cdot z_2^{i'} = z_1^i$ for some $g \in \mathrm{Sym}_k \times C_2^{\times k}$ such that either $g \cdot e_{j'} = e_j$ or $g \cdot e_{j'} = \overline{e}_j$.

(c). $ze_i \cdot ze_ie_j = ze_j \cdot ze_je_i$ if there exists the generator ze_ie_j.

We illustrate the algorithm with two easy examples. The main difficulty of the algorithm is to find $\mathcal{L}$. Once $\mathcal{L}$ is known our, it is straightforward to apply the algorithm. Example 7.2 shows that the algorithm can be applied in "pieces", each piece corresponding to an element of $\mathcal{L}$.

7.1 Example. We take $(g,p) = (1,0)$. The list $\mathcal{L}$ has a single element. We can represent the element of $\mathcal{L}$ by

$$z = (V(\Gamma), E(\Gamma), V^*(\Gamma), \mathrm{ord}) = (\{v_0, v_1, v_2\}, \{e_1, e_2, e_3, e_4\}, \{v_0^*, v_1^*, v_2^*\}, \mathrm{ord}),$$

where $\mathrm{ord}(v_0^*) = (e_1, e_2)$, $\mathrm{ord}(v_1^*) = (\overline{e}_1, e_3, e_4)$ and $\mathrm{ord}(v_2^*) = (\overline{e}_2, \overline{e}_3, \overline{e}_4)$. To simplify the notation we put

$$z = \mathrm{ord}(v_0^*); \mathrm{ord}(v_1^*), \mathrm{ord}(v_2^*) = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, \overline{e}_3, \overline{e}_4).$$

We can collapse all 4 edges of z and we have

$$\begin{aligned} z^1 &= (e_3, e_4, e_2); (\overline{e}_2, \overline{e}_3, \overline{e}_4),\\ z^2 &= (e_1, \overline{e}_3, \overline{e}_4); (\overline{e}_1, e_3, e_4),\\ z^3 &= (e_1, e_2); (\overline{e}_1, \overline{e}_4, \overline{e}_2, e_4),\\ z^4 &= (e_1, e_2); (\overline{e}_1, e_3, \overline{e}_2, \overline{e}_3). \end{aligned}$$

We see $z^1 = g^{2,1} \cdot z^2$, $z^3 = g^{4,3} \cdot z^4$, where $g^{2,1}$, $g^{4,3} \in \mathrm{Sym}_4 \times C_2^{\times 4}$ and

$$g^{2,1} = \begin{cases} e_1 & \mapsto e_3,\\ e_3 & \mapsto \overline{e}_4,\\ e_4 & \mapsto \overline{e}_2, \end{cases} \qquad \text{and} \qquad g^{4,3} = \begin{cases} e_1 & \mapsto e_1,\\ e_2 & \mapsto e_2,\\ e_3 & \mapsto \overline{e}_4. \end{cases}$$

We can collapse some edges of z^1 and z^3 and we have

$$\begin{aligned} z^{(1,3)} &= (\overline{e}_4, \overline{e}_2, e_4, e_2),\\ z^{(1,4)} &= (e_3, \overline{e}_2, \overline{e}_3, e_2),\\ z^{(1,2)} &= (e_3, e_4, \overline{e}_3, \overline{e}_4); \end{aligned} \qquad \text{and} \qquad \begin{aligned} z^{(3,1)} &= (\overline{e}_4, \overline{e}_2, e_4, e_2),\\ z^{(3,2)} &= (e_1, e_4, \overline{e}_1, \overline{e}_4). \end{aligned}$$

We see $z^{(1,2)}$, $z^{(1,3)}$, $z^{(1,4)}$, $z^{(3,1)}$ and $z^{(3,2)}$ are in the same orbit by $\mathrm{Sym}_4 \times C_2^{\times 4}$. Hence, they represent the same vertex of $(\mathrm{Sym}_4 \times C_2^{\times 4})\backslash T(z)$.

We take the maximal subtree of $(\mathrm{Sym}_4 \times C_2^{\times 4})\backslash T(z)$ with edges ze_1, ze_3 and ze_1e_2. Then $\mathcal{AM}_{1,0}$ has presentation with generators:

$$\begin{gathered} ze_1,\ ze_2,\ ze_3,\ ze_4, \\ ze_1e_3,\ ze_1e_4,\ ze_1e_2, \\ ze_2e_1,\ ze_2e_3,\ ze_2e_4, \\ ze_3e_1,\ ze_3e_2, \\ ze_4e_1,\ ze_4e_2; \end{gathered}$$

and relations:

$$\begin{aligned} & ze_1 = 1,\ ze_3 = 1,\ ze_1e_2 = 1, \\ & ze_2e_1 = ze_1e_3,\ ze_2e_3 = ze_1e_4,\ ze_2e_4 = ze_1e_2,\ ze_4e_1 = ze_3e_1,\ ze_4e_2 = ze_3e_2, \\ & ze_1 \cdot ze_1e_2 = ze_2 \cdot ze_2e_1,\ ze_1 \cdot ze_1e_3 = ze_3 \cdot ze_3e_1,\ ze_1 \cdot ze_1e_4 = ze_4 \cdot ze_4e_1, \\ & \qquad ze_2 \cdot ze_2e_3 = ze_3 \cdot ze_3e_2,\ ze_2 \cdot ze_2e_4 = ze_4 \cdot ze_4e_2. \end{aligned}$$

An easy simplification shows $\mathcal{AM}_{1,1,0} = \langle ze_2, ze_4 \mid ze_2 \cdot ze_2 = ze_4 \cdot ze_2 \cdot ze_4 \rangle$. $\square$

7.2 Example. We take $(g,p) = (0,3)$. The list $\mathcal{L}$ is

$$\begin{aligned} z_1 &= (e_1, e_2, e_3, e_4); (\overline{e}_1, e_5, \overline{e}_2), (\overline{e}_3, \overline{e}_5, \overline{e}_4), \\ z_2 &= (e_1, e_2, e_3, e_4); (\overline{e}_1, \overline{e}_4, e_5), (\overline{e}_2, \overline{e}_5, \overline{e}_3), \\ z_3 &= (e_1, \overline{e}_1, e_2, e_3); (\overline{e}_2, e_4, e_5), (\overline{e}_3, \overline{e}_5, \overline{e}_4), \\ z_4 &= (e_1, e_2, \overline{e}_2, e_3); (\overline{e}_1, e_4, e_5), (\overline{e}_3, \overline{e}_5, \overline{e}_4), \\ z_5 &= (e_1, e_2, e_3, \overline{e}_3); (\overline{e}_1, e_4, e_5), (\overline{e}_2, \overline{e}_5, \overline{e}_4), \\ z_6 &= (e_1, e_2, e_3, \overline{e}_1); (\overline{e}_2, e_4, e_5), (\overline{e}_3, \overline{e}_5, \overline{e}_4). \end{aligned}$$

For z_1 we have

$$\begin{aligned} z_1 &= (e_1, e_2, e_3, e_4); (\overline{e}_1, e_5, \overline{e}_2), (\overline{e}_3, \overline{e}_5, \overline{e}_4), \\ z_1^1 &= (e_5, \overline{e}_2, e_2, e_3, e_4); (\overline{e}_3, \overline{e}_5, \overline{e}_4), \\ z_1^2 &= (e_1, \overline{e}_1, e_5, e_3, e_4); (\overline{e}_3, \overline{e}_5, \overline{e}_4), \\ z_1^3 &= (e_1, e_2, \overline{e}_5, \overline{e}_4, e_4); (\overline{e}_1, e_5, \overline{e}_2), \\ z_1^4 &= (e_1, e_2, e_3, \overline{e}_3, \overline{e}_5); (\overline{e}_1, e_5, \overline{e}_2), \\ z_1^5 &= (e_1, e_2, e_3, e_4); (\overline{e}_1, \overline{e}_4, \overline{e}_3, \overline{e}_2). \end{aligned}$$

Generators for z_1 are:

$$\begin{gathered} z_1e_1,\ z_1e_2,\ z_1e_3,\ z_1e_4,\ z_1e_5, \\ z_1e_1e_5,\ z_1e_1e_3,\ z_1e_1e_4, \\ z_1e_2e_5,\ z_1e_2e_3,\ z_1e_2e_4, \\ z_1e_3e_1,\ z_1e_3e_2,\ z_1e_3e_5, \\ z_1e_4e_1,\ z_1e_4e_2,\ z_1e_4e_5, \\ z_1e_5e_1,\ z_1e_5e_2,\ z_1e_5e_3,\ z_1e_5e_4. \end{gathered}$$

We have

$$\begin{aligned}
z_1^{(1,5)} &= (\overline{e}_4, \overline{e}_3, \overline{e}_2, e_2, e_3, e_4),\\
z_1^{(1,3)} &= (e_5, \overline{e}_2, e_2, \overline{e}_5, \overline{e}_4, e_4),\\
z_1^{(1,4)} &= (e_5, \overline{e}_2, e_2, e_3, \overline{e}_3, \overline{e}_5),\\
z_1^{(2,5)} &= (e_1, \overline{e}_1, \overline{e}_4, \overline{e}_3, e_3, e_4),\\
z_1^{(2,3)} &= (e_1, \overline{e}_1, e_5, \overline{e}_5, \overline{e}_4, e_4),\\
z_1^{(2,4)} &= (e_1, \overline{e}_1, e_5, e_3, \overline{e}_3, \overline{e}_5),\\
z_1^{(3,5)} &= (e_1, e_2, \overline{e}_2, \overline{e}_1, \overline{e}_4, e_4),\\
z_1^{(4,5)} &= (e_1, e_2, e_3, \overline{e}_3, \overline{e}_2, \overline{e}_1).
\end{aligned}$$

We see that $z_1^{(2,4)} = g \cdot z_1^{(2,5)}$, $z_1^{(3,5)} = g' \cdot z_1^{(1,3)}$, $z_1^{(4,5)} = g'' \cdot z_1^{(1,5)}$ for some $g, g', g'' \in \mathrm{Sym}_5 \times C_2^{\times 5}$.

Relations for z_1 are:

$$\begin{aligned}
&z_1e_1 = 1,\ z_1e_2 = 1,\ z_1e_3 = 1,\ z_1e_4 = 1,\ z_1e_5 = 1,\\
&z_1e_1e_5 = 1,\ z_1e_1e_3 = 1,\ z_1e_1e_4 = 1,\ z_1e_2e_5 = 1,\ z_1e_2e_3 = 1,\\
&z_1e_1 \cdot z_1e_1e_5 = z_1e_5 \cdot z_1e_5e_1,\ z_1e_1 \cdot z_1e_1e_3 = z_1e_3 \cdot z_1e_3e_1,\\
&\qquad z_1e_1 \cdot z_1e_1e_4 = z_1e_4 \cdot z_1e_4e_1,\\
&z_1e_2 \cdot z_1e_2e_5 = z_1e_5 \cdot z_1e_5e_2,\ z_1e_2 \cdot z_1e_2e_3 = z_1e_3 \cdot z_1e_3e_2,\\
&\qquad z_1e_2 \cdot z_1e_2e_4 = z_1e_4 \cdot z_1e_4e_2,\\
&z_1e_3 \cdot z_1e_3e_5 = z_1e_5 \cdot z_1e_5e_3,\\
&z_1e_4 \cdot z_1e_4e_5 = z_1e_5 \cdot z_1e_5e_4.
\end{aligned}$$

An easy simplification shows that for z_1 generators are $z_1e_2e_4$, $z_1e_3e_5$, $z_1e_4e_5$ and for z_1 there are no relations.

From z_2 we have

$$\begin{aligned}
z_2 &= (e_1, e_2, e_3, e_4); (\overline{e}_1, \overline{e}_4, e_5), (\overline{e}_2, \overline{e}_5, \overline{e}_3),\\
z_2^1 &= (\overline{e}_4, e_5, e_2, e_3, e_4); (\overline{e}_2, \overline{e}_5, \overline{e}_3),\\
z_2^2 &= (e_1, \overline{e}_5, \overline{e}_3, e_3, e_4); (\overline{e}_1, \overline{e}_4, e_5),\\
z_2^3 &= (e_1, e_2, \overline{e}_2, \overline{e}_5, e_4); (\overline{e}_1, \overline{e}_4, e_5),\\
z_2^4 &= (e_1, e_2, e_3, e_5, \overline{e}_1); (\overline{e}_2, \overline{e}_5, \overline{e}_3),\\
z_2^5 &= (e_1, e_2, e_3, e_4); (\overline{e}_1, \overline{e}_4, \overline{e}_3, \overline{e}_2).
\end{aligned}$$

Generators for z_2 are:

$$\begin{aligned}
&z_2e_1,\ z_2e_2,\ z_2e_3,\ z_2e_4,\ z_2e_5,\\
&z_2e_1e_5,\ z_2e_1e_2,\ z_2e_1e_3,\\
&z_2e_2e_1,\ z_2e_2e_5,\ z_2e_2e_4,\\
&z_2e_3e_1,\ z_2e_3e_5,\ z_2e_3e_4,\\
&z_2e_4e_2,\ z_2e_4e_3,\ z_2e_4e_5,\\
&z_2e_5e_1,\ z_2e_5e_2,\ z_2e_5e_3,\ z_2e_5e_4.
\end{aligned}$$

We see $z_1^4 = g_{2,1}^{2,4} \cdot z_2^2$, $z_1^1 = g_{2,1}^{3,1} \cdot z_2^3$, $z_2^1 = g_{2,2}^{4,1} \cdot z_2^4$, $z_1^5 = g_{2,1}^{5,5} \cdot z_2^5$, where

$$g_{2,1}^{2,4} = \begin{cases} e_1 & \mapsto & e_1, \\ e_3 & \mapsto & \overline{e}_3, \\ e_4 & \mapsto & \overline{e}_5, \\ e_5 & \mapsto & \overline{e}_2; \end{cases} \quad g_{2,1}^{3,1} = \begin{cases} e_1 & \mapsto & e_5, \\ e_2 & \mapsto & \overline{e}_2, \\ e_4 & \mapsto & e_4, \\ e_5 & \mapsto & \overline{e}_3; \end{cases} \quad g_{2,2}^{4,1} = \begin{cases} e_1 & \mapsto & \overline{e}_4, \\ e_2 & \mapsto & e_5, \\ e_3 & \mapsto & e_2, \\ e_5 & \mapsto & e_3; \end{cases}$$

$$g_{2,1}^{5,5} = \begin{cases} e_1 & \mapsto & e_1, \\ e_2 & \mapsto & e_2, \\ e_3 & \mapsto & e_3, \\ e_4 & \mapsto & e_4. \end{cases}$$

Relations for z_2 are:

$$\begin{aligned}
&z_2e_1 = 1, z_2e_2 = 1,\\
&z_2e_2e_1 = z_1e_4e_1,\ z_2e_2e_5 = z_1e_4e_2,\ z_2e_2e_4 = z_1e_4e_5,\\
&z_2e_3e_1 = z_1e_1e_5,\ z_2e_3e_5 = z_1e_1e_3,\ z_2e_3e_4 = z_1e_1e_4,\\
&z_2e_4e_2 = z_2e_1e_5,\ z_2e_4e_3 = z_2e_1e_2,\ z_2e_4e_5 = z_2e_1e_3,\\
&z_2e_5e_1 = z_1e_5e_1,\ z_2e_5e_2 = z_1e_5e_2,\ z_2e_5e_3 = z_1e_5e_3,\ z_2e_5e_4 = z_1e_5e_4,\\
&z_2e_1 \cdot z_2e_1e_5 = z_2e_5 \cdot z_2e_5e_1,\ z_2e_1 \cdot z_2e_1e_2 = z_2e_2 \cdot z_2e_2e_1,\\
&\qquad z_2e_1 \cdot z_2e_1e_3 = z_2e_3 \cdot z_2e_3e_1,\\
&z_2e_2 \cdot z_2e_2e_5 = z_2e_5 \cdot z_2e_5e_2,\ z_2e_2 \cdot z_2e_2e_4 = z_2e_4 \cdot z_2e_4e_2,\\
&z_2e_3 \cdot z_2e_3e_5 = z_2e_5 \cdot z_2e_5e_3,\ z_2e_3 \cdot z_2e_3e_4 = z_2e_4 \cdot z_2e_4e_3,\\
&z_2e_4 \cdot z_2e_4e_5 = z_2e_5 \cdot z_2e_5e_4.
\end{aligned}$$

An easy simplification shows no new generators are needed, the relations $z_1e_4e_5 = z_1e_2e_4 \cdot z_1e_3e_5 \cdot z_1e_2e_4$, $z_1e_4e_5 \cdot z_1e_3e_5 = z_1e_2e_4 \cdot z_1e_4e_5$ are needed.

From z_3 we have

$$\begin{aligned}
z_3 &= (e_1, \overline{e}_1, e_2, e_3); (\overline{e}_2, e_4, e_5), (\overline{e}_3, \overline{e}_5, \overline{e}_4),\\
z_3^2 &= (e_1, \overline{e}_1, e_4, e_5, e_3); (\overline{e}_3, \overline{e}_5, \overline{e}_4),\\
z_3^3 &= (e_1, \overline{e}_1, e_2, \overline{e}_5, \overline{e}_4); (\overline{e}_2, e_4, e_5),\\
z_3^4 &= (e_1, \overline{e}_1, e_2, e_3); (\overline{e}_2, \overline{e}_3, \overline{e}_5, e_5),\\
z_3^5 &= (e_1, \overline{e}_1, e_2, e_3); (\overline{e}_2, e_4, \overline{e}_4, \overline{e}_3).
\end{aligned}$$

Generators for z_3 are:

$$\begin{aligned}
&z_3e_2,\ z_3e_3,\ z_3e_4,\ z_3e_5,\\
&z_3e_2e_4,\ z_3e_2e_5,\ z_3e_2e_3,\\
&z_3e_3e_2,\ z_3e_3e_5,\ z_3e_3e_4,\\
&z_3e_4e_2,\ z_3e_4e_3,\\
&z_3e_5e_2,\ z_3e_5e_3.
\end{aligned}$$

We see $z_1^2 = g_{3,1}^{2,2} \cdot z_3^2$, $z_1^2 = g_{3,1}^{3,2} \cdot z_3^3$, where

$$g_{3,1}^{2,2} = \begin{cases} e_1 & \mapsto & e_1, \\ e_3 & \mapsto & e_4, \\ e_4 & \mapsto & e_5, \\ e_5 & \mapsto & e_3; \end{cases} \quad g_{3,1}^{3,2} = \begin{cases} e_1 & \mapsto & e_1, \\ e_2 & \mapsto & e_5, \\ e_4 & \mapsto & \overline{e}_4, \\ e_5 & \mapsto & \overline{e}_3. \end{cases}$$

Relations for z_3 are:

$$\begin{aligned}
&z_3e_2,\ z_3e_4 = 1,\ z_3e_5 = 1,\\
&z_3e_2e_4 = z_1e_2e_5,\ z_3e_2e_5 = z_1e_2e_3,\ z_3e_2e_3 = z_1e_2e_4,\\
&z_3e_3e_2 = z_1e_2e_5,\ z_3e_3e_5 = z_1e_2e_3,\ z_3e_3e_4 = z_1e_2e_4,\\
&z_3e_2 \cdot z_3e_2e_4 = z_3e_4 \cdot z_3e_4e_2,\ z_3e_2 \cdot z_3e_2e_5 = z_3e_5 \cdot z_3e_5e_2,\\
&\qquad z_3e_2 \cdot z_3e_2e_3 = z_3e_3 \cdot z_3e_3e_2,\\
&z_3e_3 \cdot z_3e_3e_5 = z_3e_5 \cdot z_3e_5e_3,\ z_3e_3 \cdot z_3e_3e_4 = z_3e_4 \cdot z_3e_4e_3.
\end{aligned}$$

An easy simplification shows that neither new generators nor new relations are needed.

From z_4 we have

$$\begin{aligned}
z_4 &= (e_1, e_2, \overline{e}_2, e_3); (\overline{e}_1, e_4, e_5), (\overline{e}_3, \overline{e}_5, \overline{e}_4),\\
z_4^1 &= (e_4, e_5, e_2, \overline{e}_2, e_3); (\overline{e}_3, \overline{e}_5, \overline{e}_4),\\
z_4^3 &= (e_1, e_2, \overline{e}_2, \overline{e}_5, \overline{e}_4); (\overline{e}_1, e_4, e_5),\\
z_4^4 &= (e_1, e_2, \overline{e}_2, e_3); (\overline{e}_1, \overline{e}_3, \overline{e}_5, e_5),\\
z_4^5 &= (e_1, e_2, \overline{e}_2, e_3); (\overline{e}_1, e_4, \overline{e}_4, \overline{e}_3).
\end{aligned}$$

Generators for z_4 are:

$$\begin{aligned}
&z_4e_1,\ z_4e_3,\ z_4e_4,\ z_4e_5,\\
&z_4e_1e_4,\ z_4e_1e_5,\ z_4e_1e_3,\\
&z_4e_3e_1,\ z_4e_3e_5,\ z_4e_3e_4,\\
&z_4e_4e_1,\ z_4e_4e_3,\\
&z_4e_5e_1,\ z_4e_5e_3.
\end{aligned}$$

We see $z_1^4 = g_{4,1}^{1,4} \cdot z_4^1$, $z_1^1 = g_{4,1}^{3,1} \cdot z_4^3$, where

$$g_{4,1}^{1,4} = \begin{cases} e_2 & \mapsto & e_3, \\ e_3 & \mapsto & \overline{e}_5, \\ e_4 & \mapsto & e_1, \\ e_5 & \mapsto & e_2; \end{cases} \quad g_{4,1}^{3,1} = \begin{cases} e_1 & \mapsto & e_5, \\ e_2 & \mapsto & e_2, \\ e_4 & \mapsto & \overline{e}_4, \\ e_5 & \mapsto & \overline{e}_3. \end{cases}$$

Relations for z_4 are:

$$\begin{aligned}
&z_4e_4 = 1,\ z_4e_4 = 1,\ z_4e_5 = 1,\\
&z_4e_1e_4 = z_1e_4e_1,\ z_4e_1e_5 = z_1e_4e_2,\ z_4e_1e_3 = z_1e_4e_5,\\
&z_4e_3e_1 = z_1e_1e_5,\ z_4e_3e_5 = z_1e_1e_3,\ z_4e_3e_4 = z_1e_1e_4,\\
&z_4e_1 \cdot z_4e_1e_4 = z_4e_4 \cdot z_4e_4e_1,\ z_4e_1 \cdot z_4e_1e_5 = z_4e_5 \cdot z_4e_5e_1,\\
&\qquad z_4e_1 \cdot z_4e_1e_3 = z_4e_3 \cdot z_4e_3e_1,\\
&z_4e_3 \cdot z_4e_3e_5 = z_4e_5 \cdot z_4e_5e_3,\ z_4e_3 \cdot z_4e_3e_4 = z_4e_4 \cdot z_4e_4e_3.
\end{aligned}$$

An easy simplification shows that neither new generators nor new relations are needed.

From z_5 we have

$$\begin{aligned}
z_5 &= (e_1, e_2, e_3, \overline{e}_3); (\overline{e}_1, e_4, e_5), (\overline{e}_2, \overline{e}_5, \overline{e}_4),\\
z_5^1 &= (e_4, e_5, e_2, e_3, \overline{e}_3); (\overline{e}_2, \overline{e}_5, \overline{e}_4),\\
z_5^2 &= (e_1, \overline{e}_5, \overline{e}_4, e_3, \overline{e}_3); (\overline{e}_1, e_4, e_5),\\
z_5^4 &= (e_1, e_2, e_3, \overline{e}_3); (\overline{e}_1, \overline{e}_2, \overline{e}_5, e_5),\\
z_5^5 &= (e_1, e_2, e_3, \overline{e}_3); (\overline{e}_1, e_4, \overline{e}_4, \overline{e}_2).
\end{aligned}$$

Generators for z_5 are:

$$\begin{aligned}
&z_5e_1,\ z_5e_2,\ z_5e_4,\ z_5e_5,\\
&z_5e_1e_4,\ z_5e_1e_5,\ z_5e_1e_2,\\
&z_5e_2e_1,\ z_5e_2e_5,\ z_5e_2e_4,\\
&z_5e_4e_1,\ z_5e_4e_2,\\
&z_5e_5e_1,\ z_5e_5e_2.
\end{aligned}$$

We see $z_1^3 = g_{5,1}^{1,3} \cdot z_5^1$, $z_1^3 = g_{5,1}^{2,3} \cdot z_5^2$, where

$$g_{5,1}^{1,3} = \begin{cases} e_2 & \mapsto & \overline{e}_5, \\ e_3 & \mapsto & \overline{e}_4, \\ e_4 & \mapsto & e_1, \\ e_5 & \mapsto & e_2; \end{cases} \quad g_{5,1}^{2,3} = \begin{cases} e_1 & \mapsto & e_1, \\ e_3 & \mapsto & \overline{e}_2, \\ e_4 & \mapsto & e_5, \\ e_5 & \mapsto & \overline{e}_4. \end{cases}$$

Relations for z_5 are:

$$
\begin{aligned}
&z_5e_1 = 1,\ z_5e_4 = 1,\ z_5e_5 = 1,\\
&z_5e_1e_4 = z_1e_3e_1,\ z_5e_1e_5 = z_1e_3e_2,\ z_5e_1e_2 = z_1e_2e_4,\\
&z_5e_2e_1 = z_1e_3e_1,\ z_5e_2e_5 = z_1e_3e_4,\ z_5e_2e_4 = z_1e_3e_5,\\
&z_5e_1 \cdot z_5e_1e_4 = z_5e_4 \cdot z_5e_4e_1,\ z_5e_1 \cdot z_5e_1e_5 = z_5e_5 \cdot z_5e_5e_1,\\
&\qquad z_5e_1 \cdot z_5e_1e_2 = z_5e_2 \cdot z_5e_3e_1,\\
&z_5e_2 \cdot z_5e_2e_5 = z_5e_5 \cdot z_5e_5e_2,\ z_5e_2 \cdot z_5e_2e_4 = z_5e_4 \cdot z_5e_4e_2.
\end{aligned}
$$

An easy simplification shows that neither new generators nor new relations are needed.

From z_6 we have

$$
\begin{aligned}
z_6 &= (e_1, e_2, e_3, \overline{e}_1); (\overline{e}_2, e_4, e_5), (\overline{e}_3, \overline{e}_5, \overline{e}_4),\\
z_6^2 &= (e_1, e_4, e_5, e_3, \overline{e}_1); (\overline{e}_3, \overline{e}_5, \overline{e}_4),\\
z_6^3 &= (e_1, e_2, \overline{e}_5, \overline{e}_4, \overline{e}_1); (\overline{e}_2, e_4, e_5),\\
z_6^4 &= (e_1, e_2, e_3, \overline{e}_1); (\overline{e}_2, \overline{e}_3, \overline{e}_5, e_5),\\
z_6^5 &= (e_1, e_2, e_3, \overline{e}_1); (\overline{e}_2, e_4, \overline{e}_4, \overline{e}_3).
\end{aligned}
$$

Generators for z_6 are:

$$
\begin{aligned}
&z_6e_2,\ z_6e_3,\ z_6e_4,\ z_6e_5,\\
&z_6e_2e_4,\ z_6e_2e_5,\ z_6e_2e_3,\\
&z_6e_3e_2,\ z_6e_3e_5,\ z_6e_3e_4,\\
&z_6e_4e_2,\ z_6e_4e_3,\\
&z_6e_5e_2,\ z_6e_5e_3.
\end{aligned}
$$

We see $z_2^4 = g_{6,2}^{2,4} \cdot z_6^2$, $z_2^4 = g_{6,2}^{3,4} \cdot z_6^3$, where

$$
g_{6,2}^{2,4} = \begin{cases} e_1 & \mapsto & e_1,\\ e_3 & \mapsto & e_5,\\ e_4 & \mapsto & e_2,\\ e_5 & \mapsto & e_3; \end{cases} \qquad g_{6,2}^{3,4} = \begin{cases} e_1 & \mapsto & e_1,\\ e_2 & \mapsto & e_2,\\ e_4 & \mapsto & \overline{e}_5,\\ e_5 & \mapsto & \overline{e}_3. \end{cases}
$$

Relations for z_6 are:

$$
\begin{aligned}
&z_6e_2, z_6e_4 = 1,\ z_6e_5 = 1,\\
&z_6e_2e_4 = z_2e_4e_2,\ z_6e_2e_5 = z_2e_4e_3,\ z_6e_2e_3 = z_2e_4e_5,\\
&z_6e_3e_2 = z_2e_4e_2,\ z_6e_3e_5 = z_2e_4e_3,\ z_6e_3e_4 = z_2e_4e_5,\\
&z_6e_2 \cdot z_6e_2e_4 = z_6e_4 \cdot z_6e_4e_2,\ z_6e_2 \cdot z_6e_2e_5 = z_6e_5 \cdot z_6e_5e_2,\\
&\qquad z_6e_2 \cdot z_6e_2e_3 = z_6e_3 \cdot z_6e_3e_2,\\
&z_6e_3 \cdot z_6e_3e_5 = z_6e_5 \cdot z_6e_5e_3,\ z_6e_3 \cdot z_6e_3e_4 = z_6e_4 \cdot z_6e_4e_3.
\end{aligned}
$$

An easy simplification shows that neither new generators nor new relations are needed.

We conclude

$$\mathcal{AM}_{0,1,3} = \left\langle z_1e_2e_4,\ z_1e_3e_5,\ z_1e_4e_5 \;\middle|\; \begin{matrix} z_1e_4e_5 = z_1e_2e_4 \cdot z_1e_3e_5 \cdot z_1e_2e_4, \\ z_1e_4e_5 \cdot z_1e_3e_5 = z_1e_2e_4 \cdot z_1e_4e_5 \end{matrix} \right\rangle$$

$$= \langle z_1e_2e_4,\ z_1e_3e_5 \mid z_1e_3e_5 \cdot z_1e_2e_4 \cdot z_1e_3e_5 = z_1e_2e_4 \cdot z_1e_3e_5 \cdot z_1e_2e_4 \rangle.$$

□

Acknowledgments

I am grateful to Warren Dicks for introducing me to algebraic mapping-class groups and to Luis Paris for his advices during a post-doc in Dijon.

References

[1] Heather Armstrong, Bradley Forrest, and Karen Vogtmann. A presentation for $\mathrm{Aut}(F_n)$. *J. Group Theory*, 11(2):267–276, 2008.

[2] Heather Armstrong, Bradley Forrest, and Karen Vogtmann. A presentation for $\mathrm{Aut}(F_n)$. *J. Group Theory*, 11(2):267–276, 2008.

[3] Silvia Benvenuti. Finite presentations for the mapping class group via the ordered complex of curves. *Adv. Geom.*, 1(3):291–321, 2001.

[4] Joan S. Birman and Hugh M. Hilden. On the mapping class groups of closed surfaces as covering spaces. In *Advances in the theory of Riemann surfaces (Proc. Conf., Stony Brook, N.Y., 1969)*, pages 81–115. Ann. of Math. Studies, No. 66. Princeton Univ. Press, Princeton, N.J., 1971.

[5] Kenneth S. Brown. Presentations for groups acting on simply-connected complexes. *J. Pure Appl. Algebra*, 32(1):1–10, 1984.

[6] Marc Culler and Karen Vogtmann. Moduli of graphs and automorphisms of free groups. *Invent. Math.*, 84(1):91–119, 1986.

[7] Warren Dicks and Edward Formanek. Automorphism subgroups of finite index in algebraic mapping class groups. *J. Algebra*, 189(1):58–89, 1997.

[8] Warren Dicks and Edward Formanek. Algebraic mapping-class groups of orientable surfaces with boundaries. In *Infinite groups: geometric, combinatorial and dynamical aspects*, volume 248 of *Progr. Math.*, pages 57–116. Birkhäuser, Basel, 2005.

[9] Sylvain Gervais. A finite presentation of the mapping class group of a punctured surface. *Topology*, 40(4):703–725, 2001.

[10] John Harer. The second homology group of the mapping class group of an orientable surface. *Invent. Math.*, 72(2):221–239, 1983.

[11] A. Hatcher and W. Thurston. A presentation for the mapping class group of a closed orientable surface. *Topology*, 19(3):221–237, 1980.

[12] Susumu Hirose. A complex of curves and a presentation for the mapping class group of a surface. *Osaka J. Math.*, 39(4):795–820, 2002.

[13] Catherine Labruère and Luis Paris. Presentations for the punctured mapping class groups in terms of Artin groups. *Algebr. Geom. Topol.*, 1:73–114 (electronic), 2001.

[14] Makoto Matsumoto. A presentation of mapping class groups in terms of Artin groups and geometric monodromy of singularities. *Math. Ann.*, 316(3):401–418, 2000.

[15] James McCool. Some finitely presented subgroups of the automorphism group of a free group. *J. Algebra*, 35:205–213, 1975.

[16] James McCool. Generating the mapping class group (an algebraic approach). *Publ. Mat.*, 40(2):457–468, 1996.

[17] Bronislaw Wajnryb. A simple presentation for the mapping class group of an orientable surface. *Israel J. Math.*, 45(2-3):157–174, 1983.

[18] Bronislaw Wajnryb. Artin groups and geometric monodromy. *Invent. Math.*, 138(3):563–571, 1999.